

A CHARACTERIZATION OF ITERATIVE EQUATIONS BY THEIR COEFFICIENTS

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ABSTRACT. An expression for the coefficients of a linear iterative equation in terms of the parameters of the source equation is given both for equations in standard form and for equations in reduced normal form. The operator generating an iterative equation of a general order in reduced normal form is also obtained and some other properties of iterative equations are established. In particular, a simple necessary and sufficient condition for an equation to be iterative is given for the general fourth-order linear equation solely in terms of its coefficients.

1. INTRODUCTION

It is well-known [1] that linear ordinary differential equations (LODEs) of order one or two can all be reduced by a local diffeomorphism of the (x, y) -plane to the canonical form $y' = 0$ and $y'' = 0$, respectively, and that this is not the case for equations of a general order $n > 2$. Lie [2] showed that a differential equation of a general order $n > 2$ is equivalent (by a local diffeomorphism of the plane) to the equation $y^{(n)} = 0$, which we shall henceforth refer to as the canonical form of the linear equation, only if its symmetry algebra has the maximal dimension $n + 4$. In a much recent paper, Krausse and Michel [3] proved the converse of this statement and also showed that a LODE of order $n > 2$ has a symmetry algebra of maximal dimension if and only if it is iterative. Linear iterative equations are the iterations $\Psi^n y = 0$ of a linear first order equation, of the form

$$\Psi y \equiv r(x)y' + s(x)y = 0, \quad \Psi^n y = \Psi^{(n-1)}\Psi y. \quad (1.1)$$

In these iterations, the equation $r(x)y' + s(x)y = 0$ is termed the source equation, while the functions $r(x)$ and $s(x)$ are referred to as its parameters. Despite the unique symmetry properties of these iterative equations, many of their properties are still not known, and Mahomed [4] gave a listing of these equations for the orders three to five. We extend this list to equations of a general order, in both the standard form and the associated reduced normal form, and determine the operator generating the linear iterative equation of any given order in reduced normal form. Other properties of

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iterative equations are also established and in particular a simple criterion for a fourth-order linear equation to be iterative is given solely in terms of its coefficients.

2. EQUATIONS IN THE GENERAL LINEAR FORM

By replacing the dependent variable $y = y(x)$ by $y + y_p$, where y_p is a particular solution of the inhomogeneous equation, we may assume without loss of generality that a linear iterative equation of a general order n has the form

$$\Psi^n y \equiv K_n^0 y^{(n)} + K_n^1 y^{(n-1)} + K_n^2 y^{(n-2)} + \dots + K_n^{n-1} y' + K_n^n y = 0. \quad (2.1)$$

2.1. Case where $\Psi = \frac{d}{dx} + s$. For the sake of clarity we first consider the case where the differential operator $\Psi = rd/dx + s$ is much simpler, with $r \equiv r(x) = 1$. It is clear that in this case, the operator Ψ leaves invariant the leading coefficient, and thus we have $K_n^0 = 1$ in this case. Using the formula for $\Psi^n y$ in (1.1) gives the recurrence relations

$$K_n^1 = K_{n-1}^1 + s. \quad (2.2a)$$

$$K_n^j = K_{n-1}^j + \frac{d}{dx} (K_{n-1}^{j-1}) + s K_{n-1}^{j-1} = K_{n-1}^j + \Psi K_{n-1}^{j-1}, \quad (2.2b)$$

(for $j = 2, \dots, n-1$ and $n > 2$).

$$K_n^n = \frac{d}{dx} (K_{n-1}^{n-1}) + s K_{n-1}^{n-1} = \Psi K_{n-1}^{n-1}. \quad (2.2c)$$

Setting

$$K_m^j = 0, \quad \text{for } j < 0 \text{ or } j > m, \text{ and } K_m^j = 1, \quad \text{for } m = j = 0 \quad (2.3)$$

reduces the recurrence equations (2.2) to the single equation

$$K_n^j = K_{n-1}^j + \Psi K_{n-1}^{j-1}, \quad 0 \leq j \leq n, \quad \forall n \geq 1. \quad (2.4)$$

Solving the recurrence relations (2.4) together with the initial conditions

$$\begin{aligned} \Psi y &\equiv y' + sy = y' + K_1^1 y \\ \Psi^2 y &\equiv y'' + 2sy' + (s^2 + s')y = y'' + K_2^1 y' + K_2^2 y \end{aligned}$$

readily gives

$$\begin{aligned} K_n^1 &= ns = \binom{n}{1} \Psi^0 s \\ K_n^n &= \Psi^{n-1} s, \end{aligned}$$

for all $n \geq 1$. Using these two equalities, and setting

$$\Psi^{-1} f = 1, \quad \text{for every function } f = f(x), \quad (2.5)$$

one readily sees by induction on j and n that

$$K_n^j = \binom{n}{j} \Psi^{j-1} s, \text{ for } j = 0, \dots, n \text{ and } n \geq 1.$$

We have thus obtained the following result for the case where $r = 1$.

Theorem 1. *When the operator Ψ generating the iterative equation has the form $\Psi = d/dx + s$, that is, when it depends on the single function s , the coefficients K_n^j of the iterative equation of a general order n are given by*

$$K_n^j = \binom{n}{j} \Psi^{j-1} s, \quad \forall j = 1, \dots, n, \quad (2.6)$$

and the iterative equation of a general order n is therefore given by

$$\Psi^n y = y^{(n)} + \sum_{j=1}^n \left[\binom{n}{j} \Psi^{j-1} s \right] y^{(n-j)}. \quad (2.7)$$

Note however that (2.6) is also valid for $j = 0$.

2.2. Case where the source equation depends on both parameters.

In this case we have $\Psi = rd/dx + s$, where the parameters r , and s are given functions, and this is the most general case. Using the definition for $\Psi^n y$ given in (2.1) as well as the conventions for K_n^j set in (2.3) show that the coefficients K_n^j of the iterative equation (2.1) satisfy the recurrence relations

$$K_n^j = r K_{n-1}^j + \Psi K_{n-1}^{j-1}, \quad \text{for } 0 \leq j \leq n, \text{ and } n \geq 1, \quad (2.8)$$

which naturally reduce to (2.4) for $r = 1$. Setting $j = 0$ or $j = n$ in (2.8) readily gives by induction on n the identities

$$K_n^0 = r^n, \quad K_n^n = \Psi^{n-1} s, \quad \text{for all } n \geq 1. \quad (2.9)$$

Applying (2.8) recursively and using the conventions set in (2.3) give a new recurrence relation

$$K_n^j = \sum_{k=j}^n r^{n-k} \Psi K_{k-1}^{j-1}, \quad \text{for } j = 0, \dots, n \text{ and } n \geq 1. \quad (2.10)$$

Although Eq. (2.10) does not provide the required expression for K_n^j in terms of the parameters r and s , it represents an algorithm for the computation of the coefficients K_n^j for all possible values of n and j . For instance, using (2.10) with $j = 1, 2$ readily gives

$$K_n^1 = r^{n-1} \left[ns + \binom{n}{2} r' \right] \quad (2.11a)$$

$$K_n^2 = r^{n-2} \left[\binom{n}{2} \Psi s + \binom{n}{3} \left(3sr' + rr'' + \frac{3n-5}{4} r'^2 \right) \right]. \quad (2.11b)$$

To obtain the general expression for K_n^j using a recurrence relation relating them, we rewrite (2.10) in the form

$$K_n^j = \sum_{k=j}^n r^{(k-j)} \Psi K_{n-k+j-1}^{j-1}. \quad (2.12)$$

Then, using (2.3) and (2.9), the following formulas are successively obtained:

$$K_n^1 = \sum_{k=1}^n r^{k-1} \Psi r^{n-k} \quad (2.13a)$$

$$K_n^2 = \sum_{k_2=2}^n \sum_{k_1=1}^{n-k_2+1} r^{k_2-2} \Psi \left[r^{k_1-1} \Psi r^{n+1-(k_1+k_2)} \right] \quad (2.13b)$$

$$K_n^3 = \sum_{k_3=3}^n \sum_{k_2=2}^{n-k_3+2} \sum_{k_1=1}^{n-(k_2+k_3)+3} r^{k_3-3} \Psi \left[r^{k_2-2} \Psi \left[r^{k_1-1} \Psi r^{n+3-(k_1+k_2+k_3)} \right] \right]. \quad (2.13c)$$

Continuing this process with two more iterations by computing K_n^4 and K_n^5 , a clear pattern for the general coefficient K_n^j emerges, and to write down this expression we introduce some notations. For $n \geq 1$ and $0 \leq i \leq j \leq n$, set

$$\beta_{ij} = \sum_{u=i+1}^j k_u, \quad \text{where } k_u \in \mathbb{Z} \quad (2.14a)$$

$$M_i \equiv M_i(j) = n + \binom{j}{2} - \binom{i}{2} - \beta_{ij} \quad (2.14b)$$

$$\alpha_j = n + \binom{j}{2} - \beta_{0j} = M_0. \quad (2.14c)$$

We have the following result for the general case, where $\Psi = rd/dx + s$.

Theorem 2. *In terms of the parameters r and s of the source equation, the general coefficient K_n^j of the iterative equation (2.1) has the form*

$$K_n^j = \sum_{k_j=j}^{M_j} \sum_{k_{j-1}=j-1}^{M_{j-1}} \cdots \sum_{k_2=2}^{M_2} \sum_{k_1=1}^{M_1} r^{k_j-j} \Psi \left[r^{k_{j-1}-(j-1)} \Psi \left[\dots \Psi \left[r^{k_1-1} \Psi r^{\alpha_j} \right] \dots \right] \right], \quad (2.15)$$

for $n \geq 1$, and $1 \leq j \leq n$, and where the expressions for β_{ij} , M_i , and α_j are given by (2.14).

For the sake of clarity it would be useful to verify explicitly that (2.15) reduces indeed to (2.6) for $r = 1$.

Proposition 1. *Eq. (2.15) reduces as expected to (2.6) for $r = 1$.*

Proof. When $r = 1$, the general term in the summation (2.15) clearly reduces to $\Psi^j \cdot 1 = \Psi^{j-1}s$, and since this expression does not depend on the running indices k_1, k_2, \dots, k_j to prove the proposition, it suffices to show that the total number

$$P_{n,j} = \sum_{k_j=j}^{M_j} \sum_{k_{j-1}=j-1}^{M_{j-1}} \cdots \sum_{k_2=2}^{M_2} \sum_{k_1=1}^{M_1} 1 \quad (2.16)$$

of terms in this summation is precisely $\binom{n}{j}$. For $j = 1$, and $j = 2$, it clearly follows from (2.13a) and (2.13b) respectively that $P_{n,1} = \binom{n}{1}$, and $P_{n,2} = \binom{n}{2}$. It also follows from (2.9) that $P_{n,0}$ and $P_{n,n}$ also satisfy the required property for $n \geq 1$. Let $n \geq 2$ and assume that $P_{v,j-1} = \binom{v}{j-1}$, for $1 \leq v < n$ and $0 \leq j-1 \leq v < n$. Then it follows from (2.12) and the linearity of Ψ that $P_{n,j} = \sum_{k=j}^n \binom{n-k+j-1}{j-1}$. Setting $n-k+j-1 = q$ and $j-1 = m$ gives

$$P_{n,j} = \sum_{q=m}^{n-1} \binom{q}{m} = \binom{(n-1)+1}{m+1} = \binom{n}{j},$$

and this completes the proof by induction of the required property for $P_{n,j}$. \square

Although Proposition 1 gives a verification of the validity of the complicated formulas (2.15) at least for the simpler case $r = 1$, these formulas can be slightly simplified, by a suitable change of variables. Indeed, if in (2.15) we set

$$\begin{cases} k_j - j = n - P_j, & \text{for } j \geq 1 \\ k_i - i = P_{i+1} - P_i - 1, & \text{for } i = 1, \dots, j-1 \end{cases} \quad (2.17)$$

This reduces the expression for M_i in (2.14) to

$$M_i = P_{i+1} - 1 \quad (2.18)$$

In particular $\alpha_j = M_o$ is reduced to

$$\alpha_j = P_1 - 1 \quad (2.19)$$

Since $M_j = n$ for all j , thanks to (2.18), we may rewrite (2.17) as

$$k_i - i = M_i - P_i, \quad \text{for } i = 1, \dots, j. \quad (2.20)$$

Consequently, thanks to (2.18), (2.19) and (2.20), and after renaming the P_i , we may rewrite (2.15) in the slightly simplified form

$$K_n^j = \sum_{k_j=j}^n \sum_{k_{j-1}=j-1}^{k_j-1} \cdots \sum_{k_2=2}^{k_3-1} \sum_{k_1=1}^{k_2-1} r^{n-k_j} \Psi \left[r^{k_j-k_{j-1}-1} \Psi \left[\dots r^{k_3-k_2-1} \Psi \left[r^{k_2-k_1-1} \Psi r^{k_1-1} \right] \dots \right] \right], \quad (2.21)$$

3. EQUATIONS IN REDUCED NORMAL FORM

By diving through the general n -th order linear iterative equation $\Psi^n y$ in (2.1) by $K_n^0 = r^n$, it can be put in the form

$$y^{(n)} + B_n^1 y^{(n-1)} + \cdots + B_n^j y^{(n-j)} + \cdots + B_n^n y = 0, \quad (3.1)$$

where $B_n^j = K_n^j / r^n$. This is the standard form of the general linear iterative equation with leading coefficient one. It is well-known that (3.1) can be transformed to the normal form (in which coefficient of $y^{(n-1)}$ has vanished) by a change of the dependent variable of the form

$$y \mapsto y \exp \left(\frac{1}{n} \int_{x_0}^x B_n^1(v) dv \right). \quad (3.2)$$

However, this amounts to the requirement that $B_n^1 = 0$, i.e. that $K_n^1 = 0$. Therefore, an n -th order linear equation in reduced normal form is iterative if and only if it has the form

$$y^{(n)} + A_n^2 y^{(n-2)} + \cdots + A_n^j y^{(n-j)} + \cdots + A_n^n y = 0 \quad (3.3a)$$

where

$$A_n^j = \frac{K_n^j}{r^n} \Big|_{K_n^1=0}, \quad (2 \leq j \leq n), \quad (3.3b)$$

and where K_n^j is given by (2.15). It follows from (2.11a) that setting $K_n^1 = 0$ amounts to setting

$$s = -\frac{1}{2}(n-1)r', \quad (3.4)$$

and this shows why any iterative equation in normal form can be expressed in terms of the parameter r alone. Moreover, thanks to (3.3b) the coefficients A_n^j inherit all of the characterization (2.15) obtained for the coefficients K_n^j of iterative equations in standard form, and the corresponding expression for the A_n^j has up to the factor $1/r^n$, exactly the same form as that for the K_n^j because the parameter s does not appear explicitly in Eq. (2.15). In addition, using the same algorithm (e.g. (2.10)) obtained for the K_n^j , one can readily compute A_n^j for all $n \geq 2$ and $2 \leq j \leq n$. For instance, using the expression for K_n^2 in (2.11b) together with (3.3b), one readily sees that

$$A_n^2 = \binom{n+1}{3} A(r), \text{ where } A(r) \equiv A_2^2 = \frac{r'^2 - 2rr''}{4r^2}. \quad (3.5)$$

Theorem 3. *Let the differential operator Φ_n be given by*

$$\Phi_n = \frac{1}{r^n} \Psi^n \Big|_{K_n^1=0}. \quad (3.6)$$

Then the equation $\Phi_n y = 0$ is exactly (3.3a), that is Φ_n generates the (most general) linear iterative equation of an arbitrary order n in normal form.

Proof. Indeed, $\Psi^n y$ generates the linear iterative equation of general order n in standard form and $(1/r^n)\Psi^n$ generates the same equation with leading coefficient 1, while setting $K_n^1 = 0$ corresponds as already noted to reduce the latter equation to its normal form. \square

4. APPLICATIONS

Although formula (2.15) in Theorem 2 gives a characterization of the coefficients of a linear iterative equation of a general order in terms of the parameters of the source equation, and therefore a characterization of the linear iterative equation itself, in practice a linear equation is given solely in terms of its coefficients, and without reference to any source equation or the parameters thereof. We thus need a characterization of these iterative equations that relies solely on the coefficients of the equation. Using the operator Φ_n of Theorem 3 we can easily generate an iterative equation of a general order n , and we let the generated equation be in the form (3.3a). By analyzing the coefficients of the latter equation we readily see, at least for low order equations, that they all depend only on the coefficient A_n^2 and its derivatives. For instance, for $n = 3$ or 4 , if we set $A_3^2 = a_3$ and $A_4^2 = a_4$, then iterative equations of orders 3 and 4 take on respectively the forms

$$y''' + a_3 y' + \frac{1}{2} a_3' y = 0 \quad (4.1)$$

$$y^{(4)} + a_4 y'' + a_4' y' + \left(\frac{3}{10} a_4'' + \frac{9}{100} a_4^2 \right) y = 0 \quad (4.2)$$

Note that conversely, any equation of the form (4.1) or (4.2) is iterative, because we can always solve for r the equation $A_n^2 = \binom{n+1}{3} A(r)$ appearing in (3.5), together with (3.4) to find the parameters of the corresponding source equations. Thus each of the equations (4.1) and (4.2) characterizes the normal form of the iterative equation of the corresponding order. However, the most general characterization is to be given for equations in standard form (3.1), and the most practical characterization should be expressed explicitly in terms of the coefficients of the equation.

Let a third-order linear LODE be given in the form

$$y''' + c_2 y'' + c_1 y' + c_0 y = 0, \quad (4.3)$$

where the coefficients c_j for $j = 0, 1, 2$ are all functions of x . We may thus assume that its reduced normal form is given by (4.1). Reverting back the latter reduced equation to the corresponding equation in standard form using the inverse of the transformation of type (3.2), and dropping the

subscripts in the expression of the coefficients in (4.1) gives

$$\begin{aligned} w''' + c_2 w'' + (a + c_2' + c_2^2/3) w' \\ + \frac{1}{54} (27a' + 18c_2'' + 18ac_2 + 18c_2'c_2 + 2c_2^3) w = 0, \end{aligned} \quad (4.4)$$

where w is the new depend variable. Letting $w = y$ in (4.4) and then equating its coefficients with those of (4.3) shows that

$$a = c_1 - (c_2' + c_2^2/3) \quad (4.5a)$$

$$c_0 = \frac{1}{54} [27a' + 18c_2'' + 18ac_2 + 18c_2'c_2 + 2c_2^3]. \quad (4.5b)$$

Using (4.5a) for the expression of a and its derivatives and substituting the result in (4.5b) gives

$$54c_0 - 18c_1c_2 + 4c_2^3 - 27c_1' + 18c_2c_2' + 9c_2'' = 0. \quad (4.6)$$

The latter equation is the well-known characterization due to Laguerre [5] and Lie [6] of linear third order equations that can be reduced to the canonical form $y''' = 0$ by a point transformation. However, the methods they used to derive this conditions were different from the one used here, and which is based on the characterization of iterative equations given by (2.15) and the generating operator Φ_n in (3.6). Clearly, due to the result already cited of Krause and Michel [3] relating iterative equations and equations reducible to canonical form by point transformations, (4.6) is also a characterization of linear third order equations which are iterative.

We now move on to consider the case of a fourth order linear equation given in the form

$$y^{(4)} + c_3 y''' + c_2 y'' + c_1 y' + c_0 y = 0, \quad (4.7)$$

and we are interested in deriving necessary and sufficient conditions on the coefficients c_j ($j = 0, \dots, 3$) for the equation to be iterative. Here again, we may assume that the reduced normal form of the equation is given by (4.2). Transforming back the latter equation to its standard form, dropping the subscript in a_4 and equating the coefficients of the resulting equation with those of (4.7) gives

$$a = \frac{1}{8} (-3c_3^2 + 8c_2 - 12c_3') \quad (4.8a)$$

$$c_1 = \frac{1}{2} ac_3 + \frac{1}{16} c_3^3 + a' + \frac{3}{4} c_3 c_3' + c_3'' \quad (4.8b)$$

$$\begin{aligned} 6400c_0 = 576a^2 + 400a (4c_3' + c_3^2) + 80 (15c_3'^2 + 24a'' + 20c_3''') \\ + 1600 (a' + c_3'') c_3 + 600c_3'c_3^2 + 25c_3^4. \end{aligned} \quad (4.8c)$$

Substituting in (4.8b) and (4.8c) the expressions for a and its derivatives given by (4.8a) yields

$$0 = 4c_2c_3 - c_3^3 + 8c_2' - 6c_3c_3' - 4c_3'' - 8c_1 \quad (4.9a)$$

$$\begin{aligned} 0 = 1600c_0 - 144c_2^2 + 11c_3^4 - 400c_3c_2' + 288c_3^2c_3' + 336c_3'^2 \\ + 8c_2(c_3^2 + 4c_3') - 480c_2'' + 560c_3c_3'' + 320c_3'''. \end{aligned} \quad (4.9b)$$

We thus have the following result.

Theorem 4. *A Linear fourth order equation of the general form (4.7) is iterative if and only if its coefficients satisfy the system of two equations (4.9).*

Proof. As the system (4.9) was obtained under the assumption that (4.7) is iterative, we only need to prove conversely that the equation is iterative whenever its coefficients satisfy (4.9). The reduced normal form of (4.7) has, after the substitution of the expressions for c_0 and c_1 given by (4.9) in terms of c_2, c_3 , and their derivatives, the form

$$w^{(4)} + Q_2w'' + Q_1w' + Q_0w = 0 \quad (4.10a)$$

where

$$Q_2 = c_2 - \frac{3}{8}(c_3^2 + 4c_3') \quad (4.10b)$$

$$Q_1 = c_2' - \frac{3}{4}(c_3c_3' + 2c_3'') \quad (4.10c)$$

$$\begin{aligned} Q_0 = \frac{3}{6400}(192c_2^2 + 27c_3^4 - 48c_3'^2 - 144c_2(c_3^2 + 4c_3')) \\ + \frac{3}{6400}(27c_3^4 + 216c_3^2c_3' + 640c_2'' - 480c_3c_3'' - 960c_3'''). \end{aligned} \quad (4.10d)$$

The coefficients Q_j thus obtained clearly satisfy the conditions

$$Q_1 = Q_2' \quad \text{and} \quad Q_0 = \left(\frac{3}{10}Q_2'' + \frac{9}{100}Q_2^2\right)$$

prescribed by (4.2), and this completes the proof of the theorem. \square

5. CONCLUDING REMARKS

Some of the results that we've obtained in this paper include the expression given in (2.15) for the coefficients of an iterative equation of a general order in standard form (2.1) in terms of the parameters of the source equation, and the similar expression in (3.3b) for iterative equations in reduced normal form (3.3a). We have provided a simple algorithm (2.10) for the calculation of these coefficients for given parameters of the source equation, and the operator Φ_n obtained in Theorem 3 generates the iterative equation of an arbitrary order n in normal form. Given the already cited result of Krause and Michel [3] according to which an equation is iterative if and only

if it is reducible by a point transformation to the canonical form, the characterization of iterative equation we have given in (4.9) is also an extension to the order four of the characterization (4.6) due to Laguerre [5] and Lie [6] of linear third-order equations that are reducible to the canonical form. But unlike the characterization (4.6) which consists of a single equation, the characterization (4.9) for fourth-order equations consists of a system of two equations, and from the process of their derivation it should be expected that for equations of order n the corresponding characterization will consist of a system of $n - 2$ equations.

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